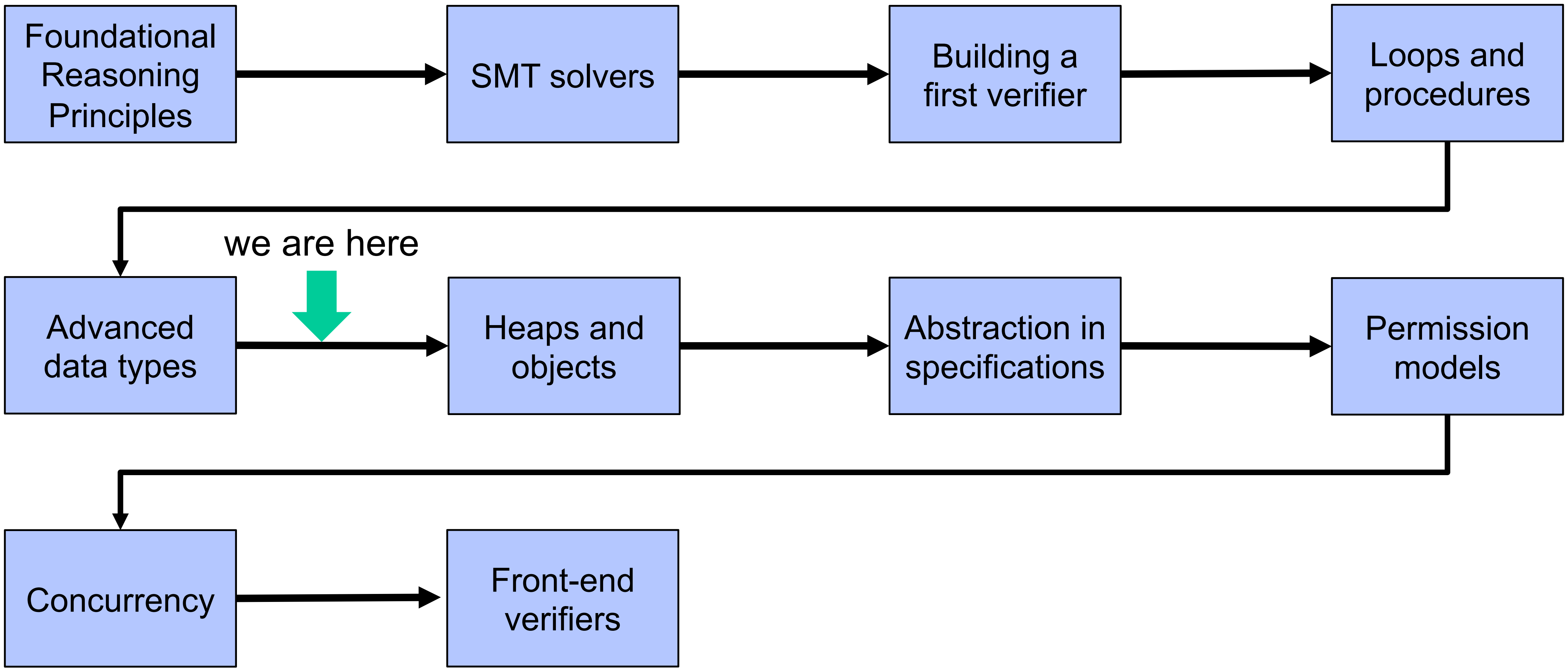
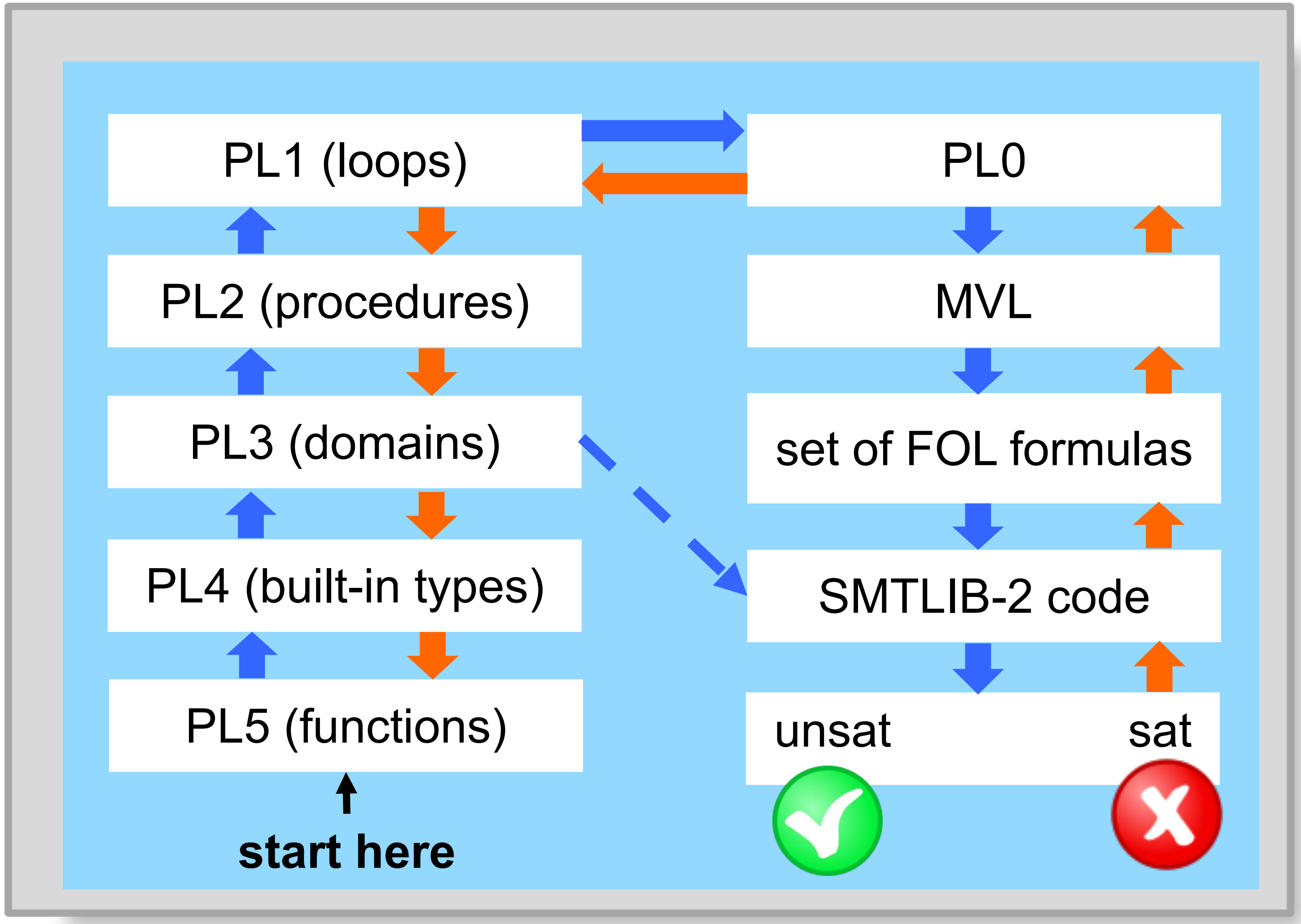
02245 – Module 6

VERIFICATION TACTICS

Tentative course outline



The language PL5

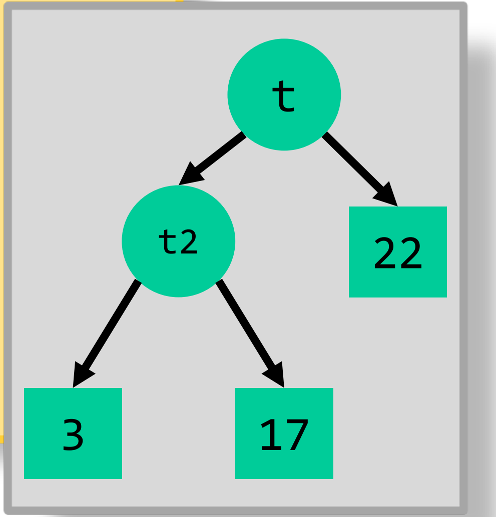


Example – summing values in a binary tree

```
method client() {
    var t: Tree := node(
      node(leaf(3), leaf(17)),
      leaf(22)
    )
    assert sum(t) == 42
}

function sum(t: Tree): Int

domain Tree {
  // ...
}
```



Previous exercise: how to define `sum`?

Try out variants: `0X-tree-sum.vpr`

Example – summing values in a binary tree

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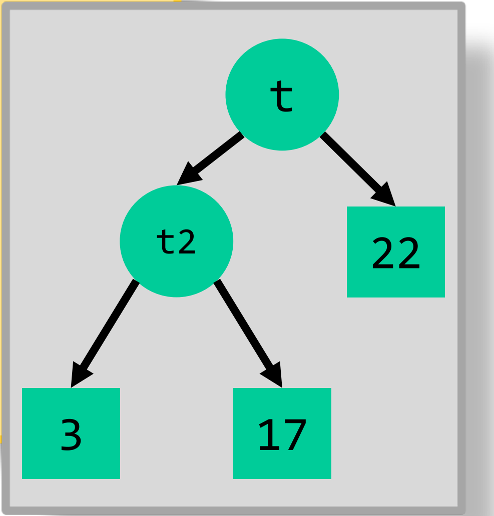
method client() {
    var t: Tree := node(
      node(leaf(3), leaf(17)),
      leaf(22)
    )
    assert sum(t) == 42
}

function sum(t: Tree): Int

domain Tree {
  // ...
}
```



Previous exercise: how to define `sum`?

Try out variants: `0X-tree-sum.vpr`

	Approach
1	implement function
2	abstract function with postcondition
3	definitional axiom using Tree functions
4	manually written definitional axioms
5	implement function + assertion in client

- All approaches are logically equivalent
- Most theories with quantifiers are undecidable

➔ To effectively use automated verifiers, we need to understand how tools deal with quantifiers

Outline: verification tactics

- Excursion: quantifiers

- Lemmas & proofs

- Hands-on program verification

Universal quantifier instantiation

- **Our problem:** Is the **FO** formula `F` unsatisfiable?
  - equivalent: is !F satisfiable?

- To prove `forall x :: G` `unsat`, we can try out all possible candidate values until we find *one value* v such that `G[x/v]` becomes `unsat`

- **Issue 1:** How do we choose *good* candidates?
  - most values may be irrelevant for our VCs

- **Issue 2:** When do we give up trying more values?
  - Logics with quantifiers are often undecidable
  - Better to quickly report that we cannot verify a problem than trying out values indefinitely

Verification condition:

```
BP && !WP(S, true)                unsat
```

```
     forall x :: G                unsat
iff  !(forall x :: G)               sat
iff  exists x :: !G                 sat
iff  for some value v, !G[x/v]      sat
iff  for some value v, G[x/v]     unsat
```

```
forall x :: G
<==>
G[x/v1] && ... && G[x/vn] &&
           && forall x :: G
```

Universal quantifier instantiation – approaches

- Due to undecidability, all approaches are incomplete
  - May return `unknown` or not terminate

- Model-based quantifier instantiation (MBQI)
  - Focuses on proving satisfiability
  - Possibly returns `unknown` instead of `unsat`
  - ➔ Not well-suited for our verification problem

Verification condition:

`BP && !WP(S, true) unsat`

- Heuristic quantifier instantiation with E-matching
  - Focuses on proving unsatisfiability
  - Possibly returns `unknown` instead of `sat`
  - We may not get a counterexample if verification fails
  - ➔ Most common approach used by verification tools

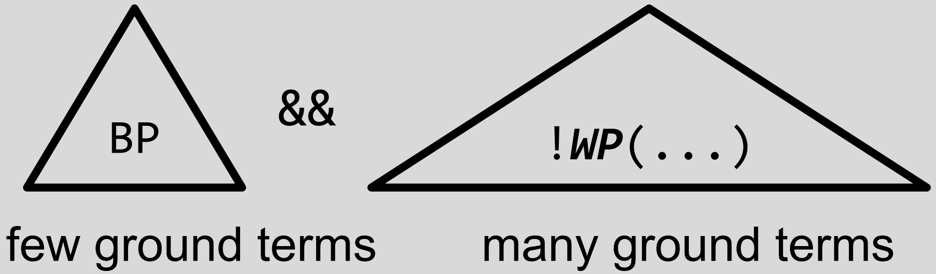
Heuristic quantifier instantiation

- Main idea: try out a subset V of all values
  - return `unsat` if `G[x/v]` is `unsat` for some v in V
  - return `unknown` if `G[x/v]` is `sat` for all v in V

```
forall x :: x in V ==> G unsat
iff for some v in V, G[x/v]  unsat
implies forall x :: G unsat
```

- Hypothesis: V should contain...
  - all ground terms
    - terms without quantifier-bound variables
    - “expressions used in program or specification”
    - E.g., `0, 1+2, x+2` (where x is a free variable)
  - function applications to ground terms
    - “unfolding of function calls”
    - E.g., `fib(fib(1))`

Structure of our VCs



- Asserting ground terms can improve quantifier instantiation ➔ `05-tree-sum.vpr`

Heuristic quantifier instantiation loop (for one quantifier)

Input: FO formula *F* `&&` **`forall`** `x :: G`

Output: `unsat` or `unknown`

```
h(0) == 1 &&
forall x :: h(x) == 1+h(x-1) && h(x) < 3
```

Algorithm:

```
F(0) :=  F
```

```
F(0) := h(0) == 1
```

```
for i = 0, 1, 2, ...
    (Choose) pick a ground term t in F(i)
    or G such that F(i) does not contain a
    conjunct equal to G[x/t]; return
    unknown if no such t exists

    (Instantiate) F(i+1) := G[x/t] && F(i)

    (Check) If F(i+1) is unsat, then
                return unsat
```

```
i = 0: choose t = 1
G[x/t] = 1 + h(0) && h(1) < 3
Instantiate: F(1) := G[x/t] && F(0)
Check: F(1) is sat ➔ continue
```

```
i = 1: choose t = 1 + h(0) = 2
G[x/t] = h(2) == 1+h(2-1) && h(2) < 3
Instantiate: F(2) := G[x/t] && F(1)
= h(2) == 1+h(2-1) && h(2) < 3
  && h(1) == 1 + h(0) && h(1) < 3
  && h(0) == 1
F(2) unsat ➔ return unsat
```

E-matching

- Problems with heuristic
  - Formulas may have exponentially many ground terms
  - Function applications admit infinitely many ground terms
    - ➔ Let user determine relevant ground terms

Predicates

```
P ::= ... | forall x:T :: { p } P
```

- A pattern (or trigger) is a term p such that
  - p contains all bound variables in the scope of the quantifier
  - p contains at least one non-constant uninterpreted function
  - p contains at most constant interpreted function

```
x == f(7) && g(x) == 3 &&
  forall y: Int ::
    { g(f(y)) } g(f(y)) > 5
```

- Consider only ground terms `t` that e-match pattern p, that is, we can find some ground term `t’` provably equal to `p[x / t]`

How can we instantiate the above?

E-matching

- Problems with heuristic
  - Formulas may have exponentially many ground terms
  - Function applications admit infinitely many ground terms
    - ➔ Let user determine relevant ground terms

Predicates

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P ::= ... | forall x:T :: { p } P
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- A pattern (or trigger) is a term p such that
  - p contains all bound variables in the scope of the quantifier
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```
x == f(7) && g(x) == 3 &&
  forall y: Int ::
    { g(f(y)) } g(f(y)) > 5
```

- Consider only ground terms `t` that e-match pattern p, that is, we can find some ground term `t’` provably equal to `p[x / t]`

`p[x/t] = g(f(7)) = 3` ← `t’` appears above
➔ instantiate `g(f(7)) > 5`

How can we instantiate the above?

Example

```
f(0) != f(1) && g(1) == 0 && f(0) == 1
&& forall x: Int ::{ p } f(x) == f(g(x))
```

- Patterns are typically terms in the quantified formulas body, e.g. `p = g(x)`
  - e-match `1`: `g(1) == 0` and `0` is a ground term
  - instantiating `f(1) == f(g(1))` makes the whole formula unsatisfiable ➔ return `unsat`
- Too restrictive patterns may often yield unknown, e.g. `p = g(g(x))`
  - No e-matching possible ➔ return `unknown`
- Too permissive patterns may lead to matching loops, e.g. `p = f(x)`
  - e-match `0`, instantiate `f(0) == f(g(0))`
  - e-match `g(0)`, instantiate `f(g(0)) == f(g(g(0)))`, e-match `g(g(0)) ...`
- Viper automatically selects (possibly suboptimal) patterns ➔ `0X-tree-sum.vpr`

Exercise

- Consider axiomatization of 2D points on the right. We added a function and an axiom for adding two points by adding their components.
- Try out different triggering patterns for the axiom on the right and test them for client below. Find patterns such that
  a) verification succeeds,
  b) verification fails, and
  c) verification does not terminate.

```
// file: examples/06-trigger-point.vpr
domain Point {
function cons(x: Int, y: Int): Point
  function first(p: Point): Int
  function second(p: Point): Int

  function add(p: Point, q: Point): Point

  axiom {
    forall p: Point, q: Point ::
     first(add(p,q)) == first(p) + first(q)
     && second(add(p,q)) == second(p) + second(q)
   }
   // ...
}
```

```
method client() {
  var x: Point := add( cons(17, 42), cons(3,8) )
  assert first(x) == 20
  assert second(x) == 50
}
```

Solution

a) verification succeeds

{ add(p,q) }

b) verification fails

`{ add(add(p,p),q) }`

c) verification does not terminate

`{ first(p), first(q) }`

```
// file: examples/06-trigger-point.vpr
domain Point {
  function cons(x: Int, y: Int): Point
  function first(p: Point): Int
  function second(p: Point): Int

  function add(p: Point, q: Point): Point

  axiom {
    forall p: Point, q: Point ::
        first(add(p,q)) == first(p) + first(q)
        &&
        second(add(p,q)) == second(p) + second(q)
   }
   // ...
}
```

Reasoning about recursive functions

➔ `07-factorial.vpr`

- Problem: Recursive functions can always be unfolded to instantiate new ground terms
- There is no natural condition for stopping the unfolding, even if the recursive function terminates
- Consequences:
  - Recursive functions lead to matching loops
  - SMT solver may never terminate
- Solution: limit the unfolding depth

```
function fac(x: Int): Int
{ x <= 1 ? 1 : x * fac(x-1) }
```

```
var n: Int; assert fac(n) != 0
```

```
function fac(x: Int): Int

axiom forall x: Int ::
 fac(x) == (x <= 1 ? 1 : x * fac(x-1))
```

```
fac(0) == 1 && fac(n) != 0
```

```
fac(1)==1 && fac(0)==1 && fac(n)!=0
```

```
fac(fac(1)) == 1 && fac(1) == 1 && ...
```

```
fac(fac(fac(1))) == 1 && ...
```

Limited functions

➔ 08-factorial.vpr

- Goal: encode recursive functions such that they can be unfolded only a limited number of times
- Idea: to stop unfolding, call a different function without a definitional axiom

```
function fac(x: Int): Int
{ x <= 1 ? 1 : x * fac(x-1) }
```

```
var n: Int; assert fac(n) != 0
```

- Viper limits recursive functions to one unfolding ➔ tree-sum-1.vpr

```
function fac(x: Int): Int
function fac0(x: Int): Int

axiom forall x: Int ::
 (x <= 1 ==> fac(x) == 1) &&
 (x > 1 ==> fac(x) == x * fac0(x-1))
```

```
fac(0) == 1 && fac(n) != 0
```

```
fac(1)==1 && fac(0)==1 && fac(n)!=0
```

```
fac(fac(1)) == fac(1) * fac0(fac(1)-1)
&& fac(1)==1 && fac(0)==1 && fac(n)!=0
```

Since `fac0` is not constrained by axioms, the SMT solver can choose a function for `fac0` such that the formula becomes `unsat`

Improving limited functions

→ `09-factorial.vpr`

- Since limited functions bound the number of unfoldings, the solver cannot find proofs that require more unfoldings
- Can we combine facts to proofs that would require multiple unfoldings?

```
  fac(1) == 1 && fac(2) == 2 * fac0(2-1)
→ fac(1) == 1 && fac(2) == 2 * fac(2-1)
→ fac(1) == 1 && fac(2) == 2 * 1
→ fac(1) == 1 && fac(2) == 2
```

```
function fac(x: Int): Int
{ x <= 1 ? 1 : x * fac(x-1) }
```

```
assert fac(1) == 1 // one unfolding
```
✓

```
assert fac(2) == 2 // two unfoldings
```
✗

```
assert fac(1) == 1
assert fac(2) == 2
// provable with one unfolding each
```
✗

Improving limited functions

- Since limited functions bound the number of unfoldings, the solver cannot find proofs that require more unfoldings

- Can we combine facts to proofs that would require multiple unfoldings?

```
  fac(1) == 1 && fac(2) == 2 * fac0(2-1)
➔ fac(1) == 1 && fac(2) == 2 * fac(2-1)
➔ fac(1) == 1 && fac(2) == 2 * 1
➔ fac(1) == 1 && fac(2) == 2
```

- Idea: axiomatize `fac(x) == fac0(x)`
  - Problem: may reintroduce matching loop

```
function fac(x: Int): Int
{ x <= 1 ? 1 : x * fac(x-1) }
```

```
assert fac(1) == 1 // one unfolding
```

```
assert fac(2) == 2 // two unfoldings
```

```
assert fac(1) == 1
assert fac(2) == 2
// provable with one unfolding each
```

```
axiom {
  forall x: Int ::
    fac(x) == fac0(x)
}
```

Improving limited functions

→ `10-factorial.vpr`

- Since limited functions bound the number of unfoldings, the solver cannot find proofs that require more unfoldings
- Can we combine facts to proofs that would require multiple unfoldings?

```
  fac(1) == 1 && fac(2) == 2 * fac0(2-1)
→ fac(1) == 1 && fac(2) == 2 * fac(2-1)
→ fac(1) == 1 && fac(2) == 2 * 1
→ fac(1) == 1 && fac(2) == 2
```

- Idea: axiomatize `fac(x) == fac0(x)`
  - Problem: may reintroduce matching loop
  - Solution: trigger axiom only for function `fac(x)`

```
function fac(x: Int): Int
{ x <= 1 ? 1 : x * fac(x-1) }
```

```
assert fac(1) == 1 // one unfolding
```

✓

```
assert fac(2) == 2 // two unfoldings
```

✗

```
assert fac(1) == 1
assert fac(2) == 2
// provable with one unfolding each
```

✓

```
axiom {
  forall x: Int :: { fac(x) }
    fac(x) == fac0(x)
}
```

Working with limited functions

→ `11-factorial.vpr`

- To work around unfolding limits, it is often sufficient to mention a ground term that is required for the proof

```
function fac(x: Int): Int
{ x <= 1 ? 1 : x * fac(x-1) }
```

```
assert fac(2) == 2
```

```
var n: Int := fac(1)
// ground term fac(1) is available
assert fac(2) == 2
```

Existential quantifier instantiation

→ `12-exists.vpr`

- **Our problem:** Is the **FO** formula `F` unsatisfiable?
  - equivalent: is !F satisfiable?

- To prove **`exists`** `x :: G` `unsat`, we have to show that `G[x/v]` becomes `unsat` for all values v

- When aiming to prove unsatisfiability, SMT solvers often struggle with existentials

```
assert exists x: Int :: x == 0
```

- Try to avoid existential quantifiers in specifications

- If needed, manually instantiate or introduce existential quantifiers → user-defined **lemmas**

Verification condition:

```
BP && !WP(S, true)                  unsat
```

```
     exists x :: G                  unsat
iff  !(exists x :: G)                 sat
iff  forall x :: !G                   sat
iff  for all values v, !G[x/v]        sat
iff  for all values v, G[x/v]       unsat
```

Outline

- Excursion: quantifiers
- Lemmas & proofs
- Hands-on program verification

Lemmas – in mathematics

- A lemma consists of
  - a premise determining whether the lemma can be used
  - a conclusion stating what property is guaranteed
  - a proof checking that the conclusion indeed always follows from the premise
- To apply a lemma, we check its premises and, if yes, can use its conclusion
- Lemmas are "subroutines of a larger proof"

Lemma 1.
Premise: `n >= 0`
Conclusion: `fac(n) > 0`
Proof: by induction on n.

```
Theorem. For all x > 0,
fac(x) + fac(x) + fac(x) > 2.

Proof.
Let x > 0. Then, since x >= 0,
Lemma 1 yields fac(x) > 0.
Hence,
fac(x) + fac(x) + fac(x) > 2.
```

Why do we need lemmas for program verification?

```
function fac(x: Int): Int {
  x <= 1 ? 1 : x * fac(x-1)
}

method client(x: Int)
  returns (y: Int)
  requires x > 0
  ensures y > 2
{
    var z: Int := fac(x)

    y := z + z + z
}
```

➔ `13-factorial-positive.vpr`

- SMT solver does not notice that `fac(x) > 0`
  - No automatic proofs by induction
  - Could be added as postcondition to `fac(x)`
- We may not want to add all needed properties as axioms (of functions)
  - Some properties might be specialized and are only useful in very specific cases
  - Many axioms might slow down proof generation

Lemmas – as ghost methods

```
method lemma(<arguments>)
  requires Premise
  ensures Conclusion
{
    Proof
}
```

```
lemma(x,y)
assert Conclusion(x,y)
```

- Lemmas are ghost methods
  - They may not affect program execution
  - They can be removed from production code
- Method body represents a correctness proof
  - Abstract methods are trusted (unproven)
- By invoking a lemma, we learn its postcondition only for the supplied the arguments

Using a lemma in Viper

➔ `14-factorial-lemma.vpr`

```
function fac(x: Int): Int {
  x <= 1 ? 1 : x * fac(x-1)
}

method client(x: Int)
  returns (y: Int)
  requires x > 0
  ensures y > 2
{
    var z: Int := fac(x)
    lemma_fac_pos(x)
    y := z + z + z
}
```

we now know `fac(x) > 0`

we do *not* know `fac(z) > 0`

```
method lemma_fac_pos(n: Int)
  requires n >= 0
  ensures fac(n) > 0
```

- By invoking a lemma, we learn its postcondition only for the supplied the arguments
- To use a lemma, we just call the method
  - Checks that premise holds for supplied arguments
  - Guarantees that conclusion holds afterward

Proving lemmas by implementing ghost methods

Statement	Meaning in proofs
`x := e`	Name an expression
`assert` `P`	Make a correct statement (possibly to introduce ground terms)
`assume` `P`	Make a (possibly wrong) assumption
`if` `(b) {S1}` **`else`** `{S2}`	Case distinction on b
method call	Invoke another lemma
recursive method call (for proofs by induction)	Invoke the induction hypothesis given by the lemma's contract

```
method lemma_fac_pos(n: Int)
  requires n >= 0
  ensures fac(n) > 0
  // decreases n // variant
{
    var v: Int := n; assert v >= 0
    // proof by induction on n
    if (n == 0) { // base case
      assert fac(0) > 0
    } else { // induction step
        assert n-1 >= 0
        // invoke I.H.
        assert n-1 < v
        lemma_fac_pos(n-1)
        assert fac(n-1) > 0
    }
}
```

➔ `15-lemma-proof.vpr`

Why we need termination proofs for lemmas

Claim: for all integers n, the following triple is valid:

```
{ n >= 0 }
lemma_fac_pos(n)
{ fac(n) > 0 }
```

Induction hypothesis (I.H.): for all m < n, the triple

```
{ m >= 0 }
lemma_fac_pos(m)
{ fac(m) > 0 }
```

is valid.

```
method lemma_fac_pos(n: Int)
  requires n >= 0
  ensures fac(n) > 0
{

    if (n == 0) {
```

induction base: claim holds for n `== 0`

```
    } else {
```

illegal use of **I.H.**: argument is not < n

```
        lemma_fac_pos(n)
    }
}
```

unsound proof!

➔ `15-lemma-proof.vpr`

Why we need termination proofs for lemmas

Claim: for all integers n, the following triple is valid:

```
{ n >= 0 }
lemma_fac_pos(n)
{ fac(n) > 0 }
```

Induction hypothesis (I.H.): for all m < n, the triple

```
{ m >= 0 }
lemma_fac_pos(m)
{ fac(m) > 0 }
```

is valid.

```
method lemma_fac_pos(n: Int)
  requires n >= 0
  ensures fac(n) > 0
{
  var v: Int := n; assert v >= 0
    if (n == 0) {
```

induction base: claim holds for n `== 0`

```
    } else {
```

illegal use of **I.H.**: argument is not < n

```
        assert n < v
        lemma_fac_pos(n)
    }
}
```

- For terminating proof programs, calls always decrease some variant

➔ only valid proofs verify

variant does not decrease
➔ unsound proof fails

➔ `15-lemma-proof.vpr`

Exercise

- Use a lemma to verify the following client:

```
// file: 16-exercise.vpr
function foo(x: Int): Int {
  x <= 0 ? 1 : foo(x - 2) + 3
}

method client(r: Int) {
  var s: Int := foo(r)
  var t: Int := foo(s)

  assert 2 <= t - r
}
```

- Bonus: prove the following lemma (including termination):

```
// file: 17-commutativity.vpr
function X(n: Int, m: Int): Int
  requires n >= 0 && m >= 0 {
  m == 0 ? 0 : n + X(n, m-1)
}

method lemma_X_commutative (n: Int, m: Int)
  requires n >= 0 && m >= 0
  ensures X(n, m) == X(m, n) {
  // TODO: show commutativity of
  //       multiplication function X
}
```

Solution

- Use a lemma to verify the following client:

```
function foo(x: Int): Int {
  x <= 0 ? 1 : foo(x - 2) + 3
}
method client(r: Int) {
  var s: Int := foo(r)
  var t: Int := foo(s)
  // ...
  assert 2 <= t - r
}
// ...
```

- Prove the following lemma (including termination):

```
function X(n: Int, m: Int): Int
  requires n >= 0 && m >= 0 {
  m == 0 ? 0 : n + X(n, m-1)
}

method lemma_X_commutative (n: Int, m: Int)
  requires n >= 0 && m >= 0
  ensures X(n, m) == X(m, n) {
    // ...
}
```

Lemmas for existential quantifiers

➔ 18-exists-lemmas.vpr

- Sometimes the solver may be unable to deal with parts of a predicate
  - Existential quantifiers, very complex predicates
- We can "hide" such predicates in a boolean function with no definitional axiom
  - The solver has nothing to unfold
  - Function calls are kept around
- We can then introduce abstract lemmas to
  - *unfold* the predicate: make its definition visible, possibly with concrete values for existentials
  - *fold* the predicate: store its definition behind a call, possibly abstracting concrete values

```
function divides(x:Int, y: Int): Bool
  requires x >= 0 && y >= 0
  //ensures result == exists z:Int ::
  //                z >= 0 && x * z == y
```

```
method divides_unfold(x: Int, y: Int)
  returns (z: Int)
  requires x > 0 && y > 0
  requires divides(x, y)
  ensures z >= 0 && x * z == y
```

```
method divides_fold(x: Int,
                    y: Int, z: Int)
  requires x > 0 && y > 0
  requires z >= 0 && x * z == y
  ensures divides(x, y)
```

Outline

- Excursion: quantifiers
- Lemmas & proofs
- Hands-on program verification

Remaining goal for today: verify programs ☺

Rules:

- There are four *verification challenges*, each with a code skeleton
  - You can work on them in groups and in any order
- Some challenges are *hard* ➔ <u>you can always ask for hints, help, or feedback</u>
- Do not modify *executable* source code unless the task is to implement something
  - You can always add ghost code, assertions, and annotations as long as you can justify their soundness (so `assume false` is not allowed, even though it might be useful on the way...).

Challenge 1: Mirroring binary trees

- Skeleton file: `challenges/mirror-tree.vpr`

- Tasks:
  a. Implement a function that mirrors binary trees, that is, swaps the left and right children <u>of every node</u> in a tree.
  b. Prove that mirroring a tree does not change the tree's size.
  c. Prove the provided client to show that mirroring an arbitrary tree twice yields the original tree.

Challenge 2: Insertion sort

- Skeleton file: `challenges/insertion-sort.vpr`

- The skeleton implements a recursive variant of insertion sort

- Tasks:
    a. Implement the function sorted stating that a sequence is sorted in ascending order.
    b. Prove the given lemma to check if your implementation is sensible.
    c. Prove (wrt. partial correctness) that the sorting method returns a sorted sequence.
    d. Prove (wrt. partial correctness) that the sorting method sorts the input sequence.
    e. Prove that the sorting method terminates.

Challenge 3: Euclid's algorithm

- Skeleton file: `challenges/euclid.vpr`

- Euclid's algorithm is a well-known iterative technique for computing the greatest common divisor (GCD) of two positive integers.

- Tasks:
  a. Define a function that returns the GCD of two positive integers.
  b. Prove that Euclid's algorithm indeed returns the GCD of its two arguments.
  c. Prove that Euclid's algorithm terminates.

Challenge 4: Determining the (clear) winner of an election

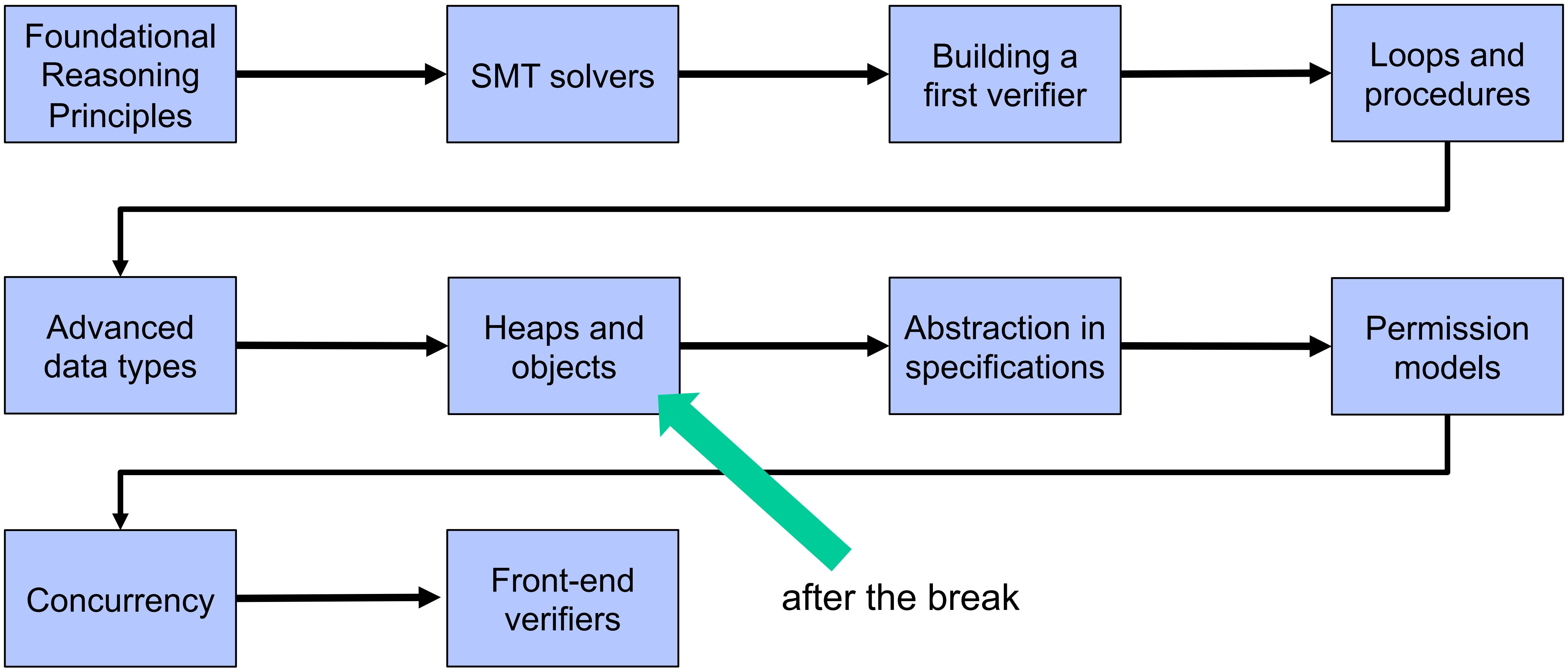
- Skeleton file: `challenges/election.vpr`

- The skeleton implements a method that takes a list of votes and attempts to return the candidate who received an absolute majority of votes (if one exists).
  - The algorithm is quite neat, since it runs in linear time.

- Tasks:
  - Verify (wrt. partial correctness) that the search method returns the winner if there is one.
    - Note: we use ghost variables to indicate who should be the winner
  - Show that the search method terminates.

< Discussion >

What next


Foundational Reasoning Principles
SMT solvers
Building a first verifier
Loops and procedures
Advanced data types
Heaps and objects
Abstraction in specifications
Permission models
Concurrency
Front-end verifiers
after the break

Feedback and muddy points

https://forms.gle/9rFHYPHFe8nGhXLx5