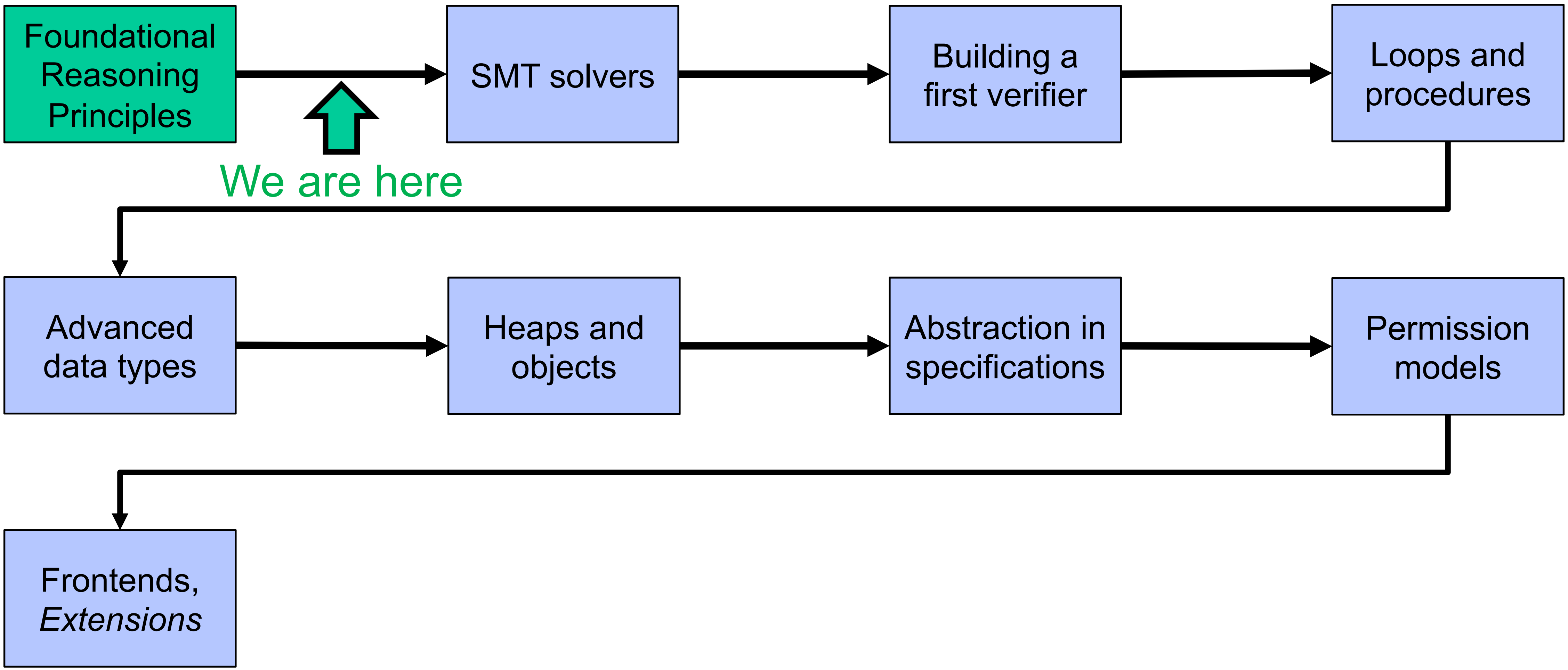
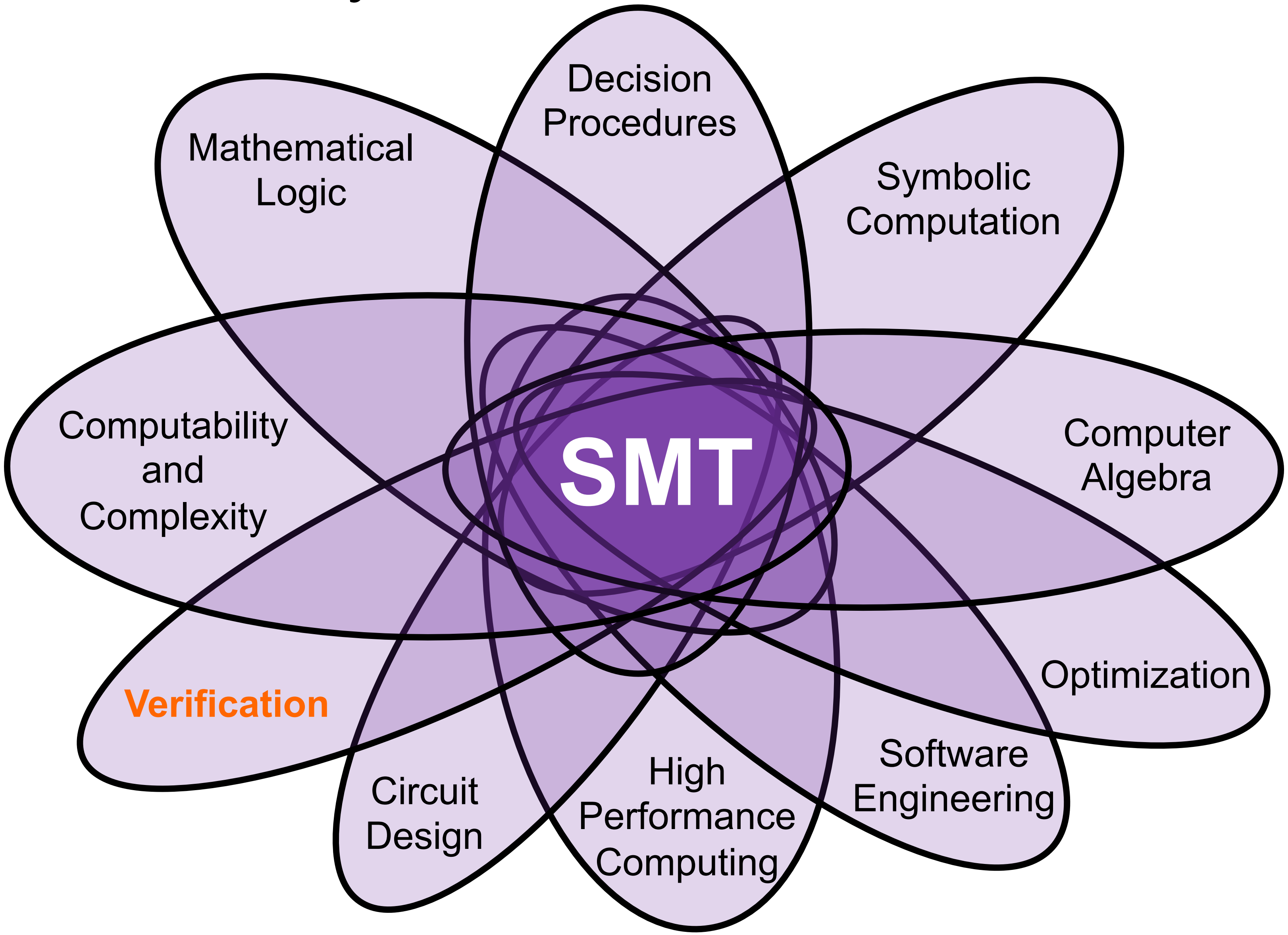
02245 – Lecture 2

FOUNDATIONS & SMT SOLVERS

Tentative course outline



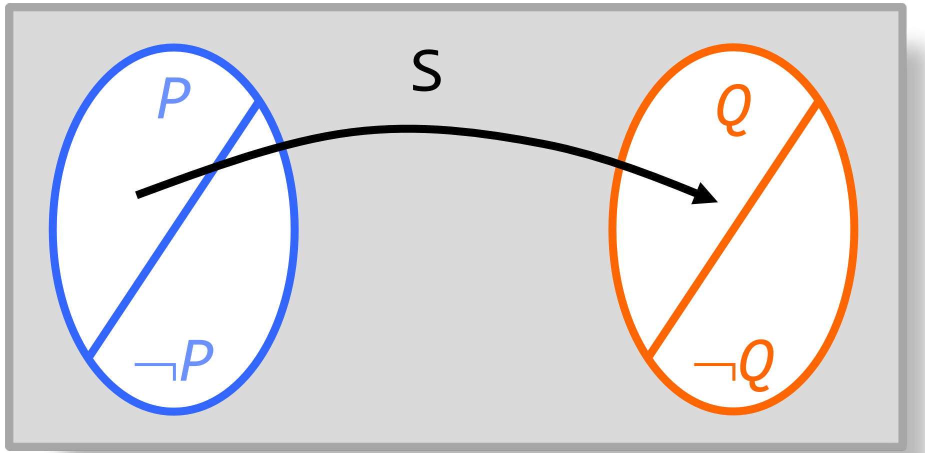
Satisfiability Modulo Theories Solvers



- A foundational topic in theoretical and applied computer science
- Our focus:
  effectively applying SMT technology to program verification

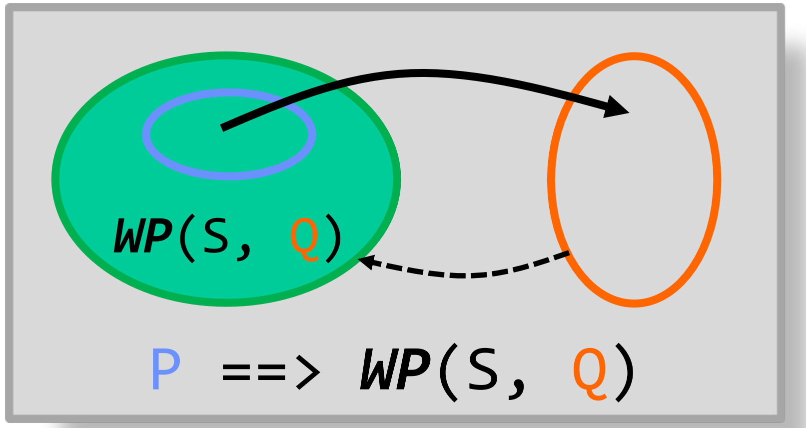
But first: Recap

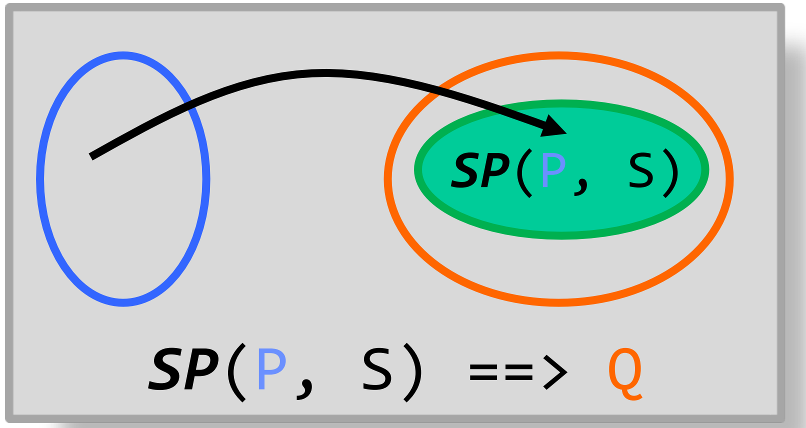
The **Floyd-Hoare triple** { *P* } S { *Q* } is **valid** if and only if
every execution of S that starts in a state satisfying precondition *P*
terminates without an error in a state satisfying postcondition *Q*.




```
method foo(x: Int)
  returns (r: Int)
  requires x > 0
  ensures r > y
{
  // S
  var y: Int := 7
  r := x + y
}
```

Recap: Weakest Pre & Strongest Post





S	WP(S, Q) (total correctness)	SP(P, S) (partial correctness: accepts errors/divergence)
var x	forall x :: Q	exists x :: Q
x := a	Q[x / a]	exists x0 :: P[x / x0] && x == a[x / x0]
assert R	R && Q	P && R
assume R	R ==> Q	P && R
S1; S2	WP(S1, WP(S2, Q))	SP(SP(P, S1), S2)
S1 [] S2	WP(S1, Q) && WP(S2, Q)	SP(P, S1) || SP(P, S2)

Automating Program Verification

Mains steps of a tool for checking that `{ P } S { Q }` is valid:

1. Compute `WP(S, Q)` → last lecture

2. Check whether `P ==> WP(S, Q)` is valid → delegate to SMT solver

Alternative approach

Mains steps of a tool for checking that `{ P } S { Q }` is valid:

1. Compute ***SP***`(P, S)` and ***SAFE***`(P, S)`

2. Check whether ***SP***`(P, S) ==> Q` is valid

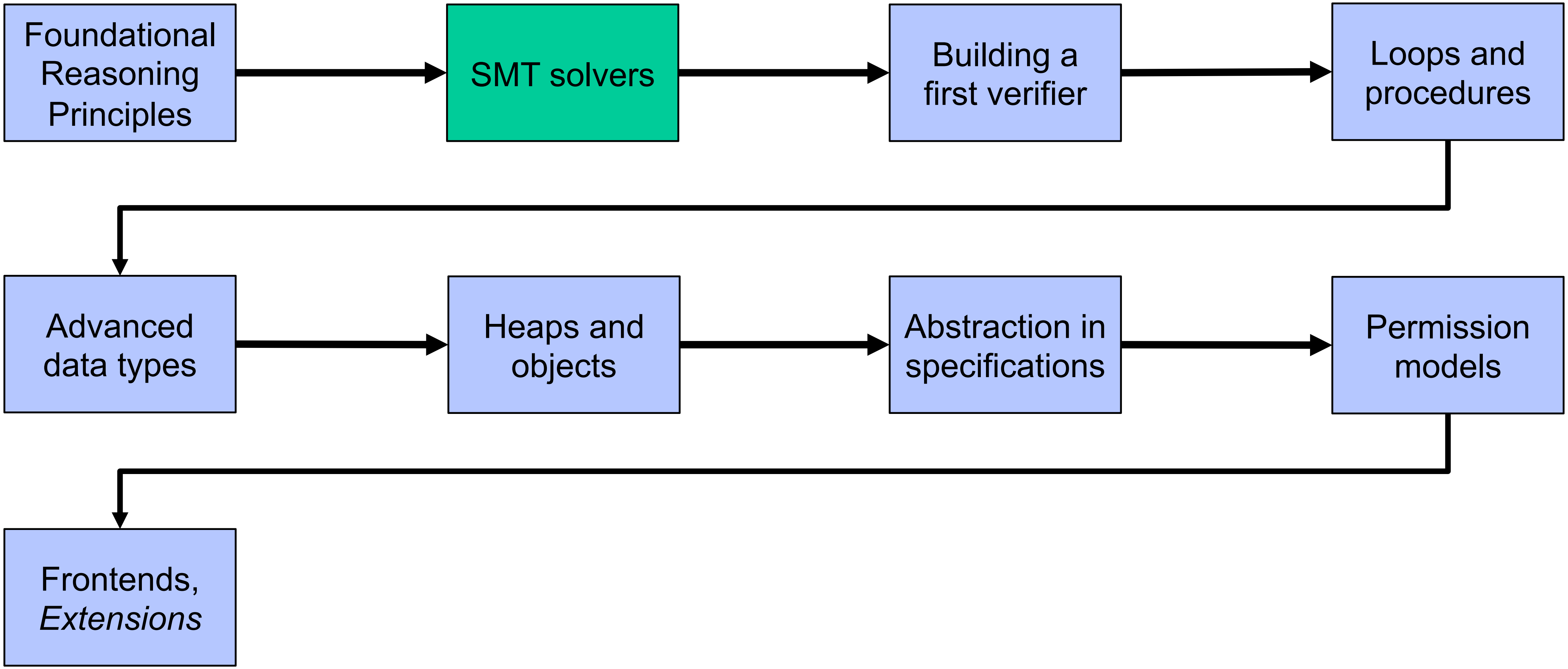
   and ***SAFE***`(P, S)` is valid

➔ Homework

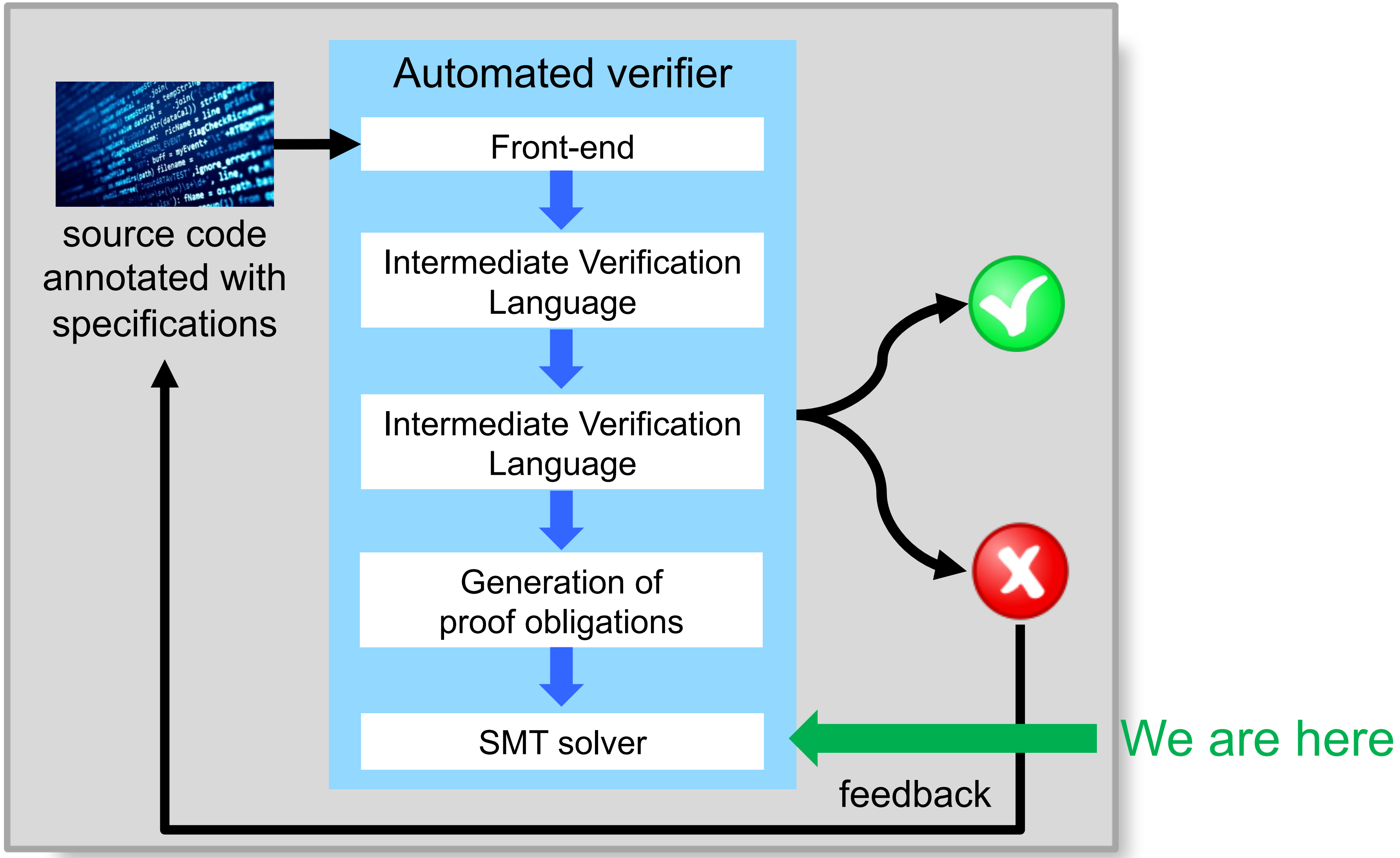
< Homework W1 >

Solutions will be published on course page

Tentative course outline



Roadmap



Automated verifier
Front-end
Intermediate Verification Language
Intermediate Verification Language
Generation of proof obligations
SMT solver
source code annotated with specifications
feedback
We are here

Overview

1. Propositional logic and SAT solvers
2. Using Z3 as a SAT solver
3. First-order logic and SMT solvers
4. Using Z3 as an SMT solver

Propositional Logic

X: Boolean variable in **Var**

Syntax

$F ::= \texttt{false} \mid \texttt{true} \mid X \mid \neg F \mid F \land F \mid F \lor F \mid F \Rightarrow F \mid F \Leftrightarrow F$

Interpretation $\mathfrak{I}: \mathbf{Var} \to \{\text{true}, \text{false}\}$

Satisfaction relation

$\mathfrak{I} \vDash \texttt{true}$ iff always

$\mathfrak{I} \vDash X$ iff $\mathfrak{I}(X) = \text{true}$

$\mathfrak{I} \vDash \neg F$ iff not $\mathfrak{I} \vDash F$

$\mathfrak{I} \vDash F \land G$ iff $\mathfrak{I} \vDash F$ and $\mathfrak{I} \vDash G$

$\mathfrak{I}$ is a **model** of F iff $\mathfrak{I} \vDash F$

$\mathfrak{I} ::= [X = \text{false}, Y = \text{true}]$

$\mathfrak{I} \vDash \neg X \lor Y$

$\mathfrak{I} \vDash X \Rightarrow Y$

$\mathfrak{I} \vDash (\neg X \lor Y) \Leftrightarrow (X \Rightarrow Y)$

Satisfiability & Validity

- **F** is satisfiable iff **F** has some model

$(X \Rightarrow Y) \Rightarrow Y$

Models: $[X = \text{true}, Y = \text{true}]$, $[X = \text{false}, Y = \text{true}]$, $[X = \text{true}, Y = \text{false}]$

- **F** is unsatisfiable iff **F** has no model

$X \land \neg Y \land (X \Rightarrow Y)$

- **F** is valid iff every interpretation is a model of **F** ($\neg$**F** is unsatisfiable)

$X \land (X \Rightarrow Y) \Rightarrow Y$

- **F** is not valid iff some interpretation is not a model of **F** ($\neg$**F** is satisfiable)

$X \land (X \Rightarrow Y) \Leftrightarrow Y$

Model of $\neg$**F**: $[X = \text{false}, Y = \text{true}]$

The Satisfiability Problem

- A formula is satisfiable if it has a model

Satisfiability Problem (SAT):
Given a propositional logic formula, decide whether it is satisfiable.

- If yes, provide a model as a witness

$$(X \lor Y \lor \neg Z) \land (U \lor \neg Y) \land (\neg X \lor \neg Z \lor U \lor V)$$

$$\mathfrak{I} ::= [\ U = \text{false},\ V = \text{false},\ X = \text{true},\ Y = \text{false},\ Z = \text{false}\ ]$$

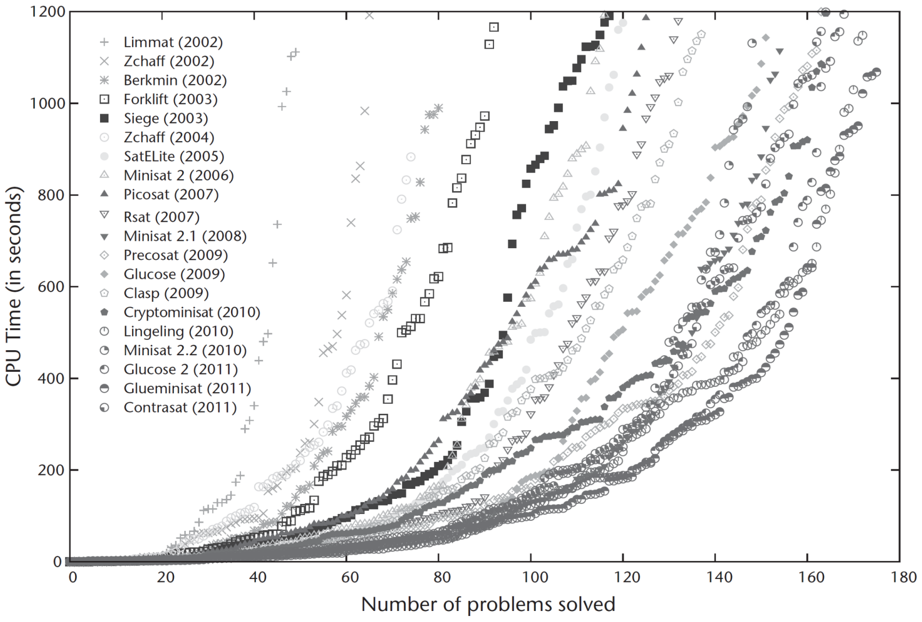
Complexity of SAT

- For formulas in conjunctive normal form (CNF), SAT is the classical NP-complete problem

$$\bigwedge_i \bigvee_j C_{i,j} \quad \text{where } C_{i,j} \in \{X_{i,j}, \neg X_{i,j}\}$$

- Many difficult problems can be efficiently encoded

- Every known algorithm is exponential in the formula's size

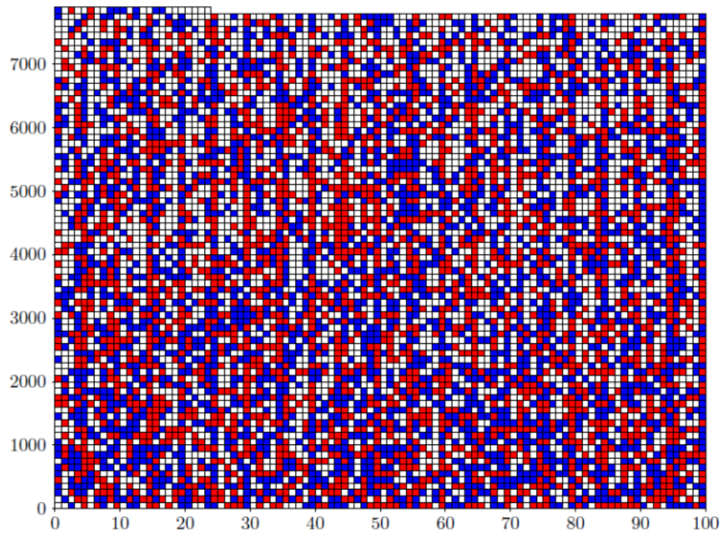


Example: Boolean Pythagorean Triples

- BPT: a triple of natural numbers $1 \leq a \leq b \leq c$ with $a^2 + b^2 = c^2$
- Question: Can we color all natural numbers with just two colors such that no BPT is monochromatic?
- Answer: No! The set {1, ..., 7825} always contains a monochromatic BPT

- This was first proven using a SAT solver
  - number of combinations: 2^{7825}
  - "the largest math proof ever" (ca. 200 TB)

- **Modern SAT solvers are efficient in practice**



credits: Marijn J.H. Heule, "Everything's Bigger in Texas - The Largest Math Proof Ever", GCAI 2017

Exercise: Seating of Wedding Guests

Exercise

- Model the following problem as an instance of the SAT problem.

- There are three chairs in a row: left, middle, right.

- Can we assign chairs to Alice, Bob, and Charlie such that:
  - Alice does not sit next to Charlie,
  - Alice does not sit on the leftmost chair, and
  - Bob does not sit to the right of Charlie?

Solution

- Model assignment via nine variables

$x_{p,c}:$ "person p sits in chair c"

- Alice does not sit next to Charlie

$(x_{A,l} \vee x_{A,r} \Rightarrow \neg x_{C,m}) \wedge (x_{A,m} \Rightarrow \neg x_{C,l} \wedge \neg x_{C,r})$

- Alice does not sit on the leftmost chair

$\neg x_{A,l}$

- Bob does not sit to the right of Charlie

$(x_{C,l} \Rightarrow \neg x_{B,m}) \wedge (x_{C,m} \Rightarrow \neg x_{B,r})$

- Each person gets a chair

$$\bigwedge_{1 \leq p \leq 3} \bigvee_{1 \leq c \leq 3} x_{p,c}$$

- Every person gets at most one chair

$$\bigwedge_{1 \leq p \leq 3} \bigwedge_{1 \leq c,d \leq 3, c \neq d} (\neg x_{p,c} \vee \neg x_{p,d})$$

- Every chair gets at most one person

$$\bigwedge_{1 \leq p,q \leq 3, p \neq q} \bigwedge_{1 \leq c \leq 3} (\neg x_{p,c} \vee \neg x_{q,c})$$

Overview

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The Z3 Satisfiability Modulo Theories solver

- Developed by Microsoft (under MIT license)

- Building block of many verification tools including Viper

- Various input formats and APIs
  - Z3, SMTLIB-2, C, C++, Python, Java, Rust, OCaml, ...

- For now: Use Z3 as a SAT solver

A first example (SMTLIB-2)

```
; declare variables
(declare-const X Bool)
(declare-const Y Bool)
(declare-const Z Bool)

; define formula (X ⇒ Y ⇒ Z) ∧ X
(assert (=> X Y Z))
(assert X)

(check-sat)

(get-model) ; fails if unsat
```

```
$ z3 01-example.smt2

sat
(model
  (define-fun Z () Bool
    false)
  (define-fun X () Bool
    true)
  (define-fun Y () Bool
    false)
)
```

A first example (Z3Py)

```
from z3 import *

# declare variables
X = Bool('X')
Y = Bool('Y')
Z = Bool('Z')

# define formula F
F = And( Implies(X, Implies(Y, Z)), X)

solve(F) # find a model for F

# find a counterexample for F
solve(Not(F))
```

F is satisfiable, this is a model

```
$ python .\02-example.py
[Z = False, X = True, Y = False]
[Z = False, X = False, Y = True]
```

$\neg$**F** is satisfiable, this is a model

Example: Course Selection

- You have to take CS Modeling, Physics, or Chemistry
- For CS Modeling, you also need Discrete Math
- For Verification, you need CS Modeling
- For Physics and Chemistry, you need Statistics
- Statistics and Discrete Math are at the same time
- CS Modeling and Physics are at the same time
- Verification and Chemistry are at the same time

Is it possible to take Verification and all preliminaries?

Is it possible to take Physics and Discrete Math?

Solution: Course Selection

```
(declare-const Verification Bool)
; ...
```

```
(assert
    (and
        (or ComputerScienceModelling Physics Chemistry)
        (=> ComputerScienceModelling DiscreteMath)
        (=> Verification ComputerScienceModelling )
        (=> (or Physics Chemistry) Statistics)
        (xor Statistics DiscreteMath)
        (xor ComputerScienceModelling Physics)
        (xor Verification Chemistry)
    )
)
```

```
(assert Verification)
(check-sat)
(get-model)
```

Exercise: Seating of Wedding Guests

- Use Z3 to check whether we can assign suitable seats to all wedding guests

- There are three chairs in a row: left, middle, right.

- We want to assign chairs to Alice, Bob, and Charlie such that:
  - Alice does not sit next to Charlie,
  - Alice does not sit on the leftmost chair, and
  - Bob does not sit to the right of Charlie.

Solutions in example files

04-wedding.smt2 / .py

Proofs with Z3

```
(declare-const x Bool)
(declare-const y Bool)

(echo "De Morgan's law: !(x && y) == (!x || !y)")
(assert
    (=
        (not (and x y) )
        (or (not x) (not y))
    )
)
(check-sat) ; result: sat
```

What does this tell us about De Morgan's law?

Proofs with Z3

```
(declare-const x Bool)
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(echo "De Morgan's law: !(x && y) == (!x || !y)")
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    (=
        (not (and x y) )
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    )
)
(check-sat) ; result: sat
```

there is an example for which the law is true

What does this tell us about De Morgan's law?

Proofs with Z3

```
(declare-const x Bool)
(declare-const y Bool)

(echo "De Morgan's law: !(x && y) == (!x || !y)")
(assert
  (not
    (=
        (not (and x y) )
        (or (not x) (not y))
    )
  )
)
(check-sat) ; result: unsat
```

What does this tell us about De Morgan’s law?

Proofs with Z3

```
(declare-const x Bool)
(declare-const y Bool)

(echo "De Morgan's law: !(x && y) == (!x || !y)")
(assert
  (not
    (=
        (not (and x y) )
        (or (not x) (not y))
    )
  )
)
(check-sat) ; result: unsat
```

There is no counterexample

➔ the formula is valid

➔ De Morgan’s law holds

What does this tell us about De Morgan’s law?

Using Z3 for Homework

- Here is an excerpt from a proof in the first homework assignment:

```
valid: P ==> WP(assert R, Q)
iff

valid: P ==> (R && Q)
iff
valid: P ==> R
and valid: (P && R) ==> Q

iff
valid: SAFE(p, assert R)
and valid: SP(P, assert R) ==> Q
```

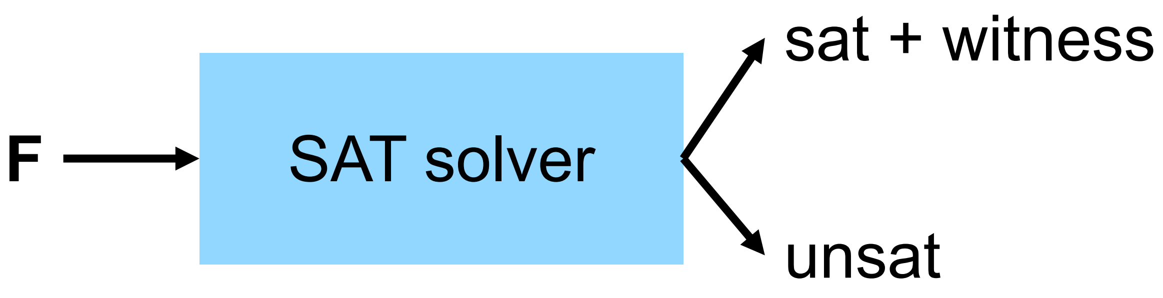
- Use Z3 to prove the blue equivalence.

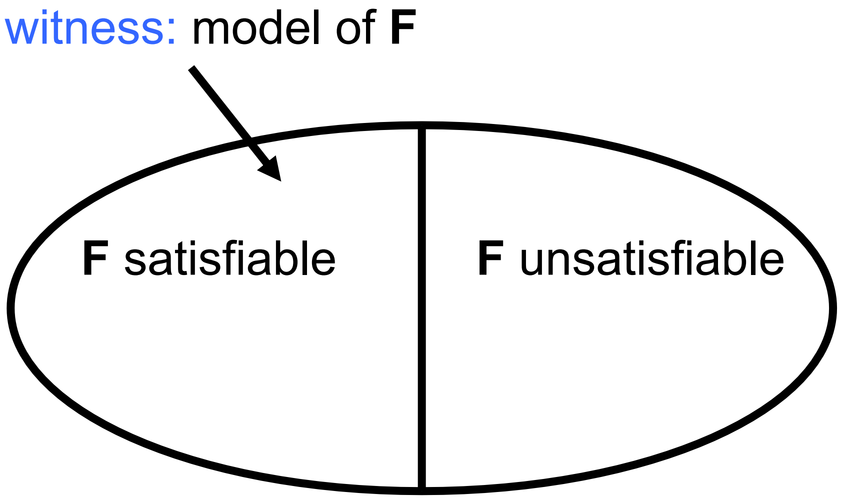
Solution

see code examples

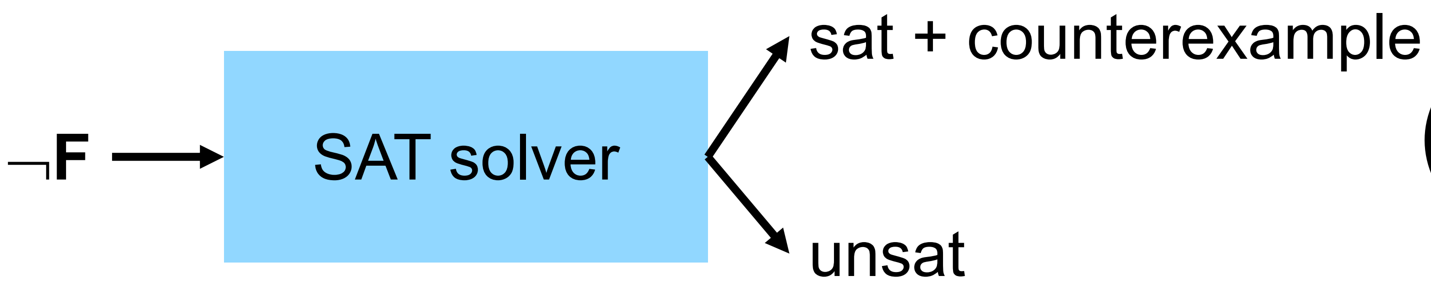
Using a SAT solver

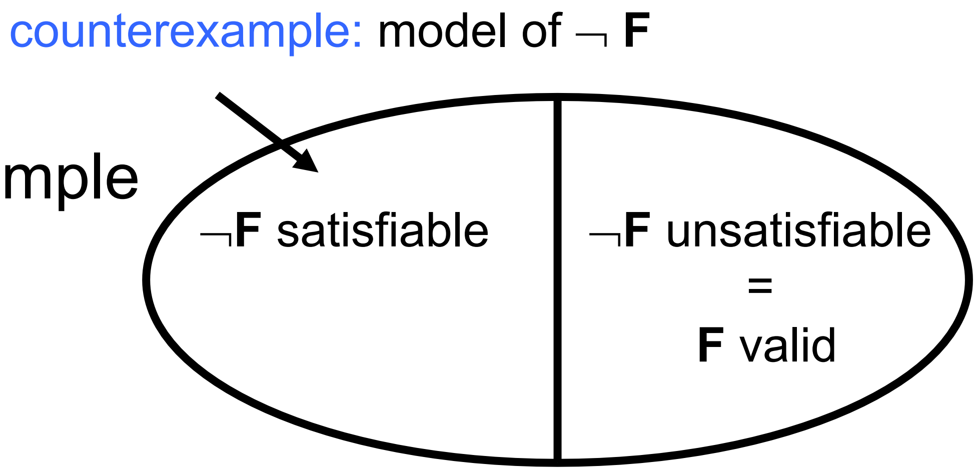
- Is **F** satisfiable?





- Is **F** valid?





Using a SAT Solver for Program Verification

Mains steps of a tool for checking that `{ P } S { Q }` is valid:

1. Compute `wp(S, Q)` → last lecture

2. Check whether entailment `P ==> WP(S, Q)` is valid → ask SAT solver

   - Check satisfiability of negation: `P && !WP(S, Q)`
   - `unsat` → `{ P } S { Q }` is valid
   - `sat` → model explains why `{ P } S { Q }` is not valid

Using a SAT Solver for Program Verification

```
{ true }
// check that validity of true ⇒ a ∧ ...
{ a ∧ (b ∧ (true ⇔ (a ⇒ b)) ∨ ¬b ∧ (false ⇔ (a ⇒ b))) ∨ ¬a ∧ (true ⇔ (a ⇒ b)  }
if (a) {
{ b ∧ (true ⇔ (a ⇒ b)) ∨ ¬b ∧ (false ⇔ (a ⇒ b)) }
  if (b) {
{ true ⇔ (a ⇒ b) }
    res := true
{ res ⇔ (a ⇒ b) }
  } else {
{ false ⇔ (a ⇒ b) }
    res := false
{ res ⇔ (a ⇒ b) }
  }
{ res ⇔ (a ⇒ b) }
} else {
{ true ⇔ (a ⇒ b) }
  res := true
{ res ⇔ (a ⇒ b) }
}
{ res ⇔ (a ⇒ b) }
```

Propositional logic is not enough

```
{ x == X && y == Y }
{ y == Y && y - Y == 0 }
// ... swap X and Y
{ x == Y && y == X }
```

Entailment to check:

```
(x == X && y == Y) ==> y == Y && y - Y == 0
```

- Entailment is not in propositional logic
  - Integer-valued variables (`x,X,y,Y`) and numeric constants (0)
  - Arithmetic operations (`-`) and comparisons (`==`)
- Logic must support at least the expressions appearing in programs
  - It is also useful to support quantifiers (e.g., for array algorithms)
- General framework: first-order predicate logic (FOL)

Overview

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Ingredients of Many-sorted First-order logic (FOL)

1. Sorts — `Bool, Int, Real, T`
   - specifies possible types
2. Typed Variables — `x, y, z, …`
3. Typed Function symbols — `0, 1.5`   `+, *, _?_:_`
   - building blocks of terms — `0`   `x ? y - 17 : z*z + 1`
4. Typed Relational symbols — `=`   `<`   `prime`   `R`
   - turn terms into logical propositions — `x = 0`   `prime(y+4)`   `R(x,y,z)`
5. Logical symbols — $\land$ $\lor$ $\neg$ $\Rightarrow$ $\Leftrightarrow$ $\exists$ $\forall$ …

FOL Formulas

- A **signature Σ** is a set of
  - at least one sort
  - function symbols
  - relational symbols (including $=$)

$$\Sigma = \{\, \mathbf{Int}, 0, 1, +, *, <, = \,\}$$

- A **Σ-formula** is a logical formula over propositions built from symbols in Σ

$$\forall x{:}\, \mathbf{Int}\ \exists y{:}\, \mathbf{Int}\ (y = x + 1 \land y * y = x * x + (1 + 1) * x + 1)$$

Is this Σ-formula satisfiable?

FOL Formulas

Is this Σ-formula satisfiable?

$$\Sigma = \{\ \mathbf{Int}, 0, 1, +, *, <, =\ \}$$

$$\forall x{:}\ \mathbf{Int}\ \exists y{:}\ \mathbf{Int}\ (y = x + 1 \land y * y = x * x + (1 + 1) * x + 1)$$

Yes, if the symbols $+, *, =$ have the canonical meaning

No, if

- 1 actually means 2, or
- \+ actually means maximum

Satisfiability of Σ-formulas depends on the admissible interpretations of symbols in Σ

determined by Σ-theories

FOL Σ-Interpretations

$\Sigma = \{ \mathbf{Int}, one, plus, eq \}$

- A **Σ-structure $\mathfrak{A}$** assigns
  - a non-empty domain (set) $\mathbf{U}^{\mathfrak{A}}$ to each sort $\mathbf{U}$ in Σ
  - a function $f^{\mathfrak{A}}$ over domains (respecting types) to each function symbol f in Σ
  - a relation $R^{\mathfrak{A}}$ over domains (respecting types) to each relational symbol R in Σ

$\mathfrak{A} ::= (\mathbf{Int}^{\mathfrak{A}}, one^{\mathfrak{A}}, plus^{\mathfrak{A}}, eq^{\mathfrak{A}})$

$\mathbf{Int}^{\mathfrak{A}} ::= \mathbb{Z}$

$one^{\mathfrak{A}} ::= 1$

$plus^{\mathfrak{A}}: \mathbf{Int}^{\mathfrak{A}} \times \mathbf{Int}^{\mathfrak{A}} \to \mathbf{Int}^{\mathfrak{A}}, (a,b) \mapsto a + b$

$eq^{\mathfrak{A}} ::= \{ (a,b) \in \mathbf{Int}^{\mathfrak{A}} \times \mathbf{Int}^{\mathfrak{A}} \mid a = b \}$

- A **Σ-assignment $\boldsymbol{\beta}$** maps variables x of sort $\mathbf{U}$ to domain elements in $\mathbf{U}^{\mathfrak{A}}$

$\beta: \mathbf{Var} \to \mathbf{Int}^{\mathfrak{A}}$

- A **Σ-interpretation** is a pair $\mathfrak{I} = (\mathfrak{A}, \beta)$

- $\mathfrak{I}(t)$ denotes the domain element obtained by evaluating term t in $\mathfrak{I}$

$\mathfrak{I}(plus(plus(one, one), x))$
$= \mathrm{plus}^{\mathfrak{A}}(\mathrm{plus}^{\mathfrak{A}}(\mathrm{one}^{\mathfrak{A}}, \mathrm{one}^{\mathfrak{A}}), \beta(x))$
$= (1+1) + \beta(x)$

FOL Semantics

$\mathfrak{I}$ is a model of **F** iff $\mathfrak{I} \vDash$ **F**

FOL formula F (excerpt)	$\mathfrak{I} = (\mathfrak{A}, \beta) \vDash$ F if and only if
$t_1 = t_2$	$\mathfrak{I}(t_1) = \mathfrak{I}(t_2)$
$R(t_1, \ldots, t_n)$	$(\mathfrak{I}(t_1), \ldots, \mathfrak{I}(t_n)) \in R^{\mathfrak{A}}$
$\mathbf{G} \land \mathbf{H}$	$\mathfrak{I} \vDash \mathbf{G}$ and $\mathfrak{I} \vDash \mathbf{H}$
$\mathbf{G} \Rightarrow \mathbf{H}$	If $\mathfrak{I} \vDash \mathbf{G}$, then $\mathfrak{I} \vDash \mathbf{H}$
$\exists x \colon \mathbf{T}\ (\mathbf{G})$	For some $\mathrm{v} \in \mathbf{T}^{\mathfrak{A}}$, $\mathfrak{I}[x := \mathrm{v}] \vDash \mathbf{G}$
$\forall x \colon \mathbf{T}\ (\mathbf{G})$	For all $\mathrm{v} \in \mathbf{T}^{\mathfrak{A}}$, $\mathfrak{I}[x := \mathrm{v}] \vDash \mathbf{G}$

F is satisfiable iff **F** has some model

Issues with FOL Satisfiability

$\Sigma = \{ \mathbf{Nat}, zero, one, plus, eq \}$

- All symbols are uninterpreted

- The meaning of functions and relations is determined in the chosen model

$\mathbf{F} ::= \exists x : \mathbf{Nat}(\, x \; plus \; one \; eq \; zero)$

Infix notation for $eq(plus(x, one), zero)$

- Many formulas are satisfiable if we can choose Σ-structures that defy the intended meaning of functions and relations

sat: $\mathbf{Nat} ::= \mathbb{N}, one^{\mathfrak{A}} ::= 0, eq ::= =$
$zero^{\mathfrak{A}} ::= 1, plus^{\mathfrak{A}} ::= +$

sat: $\mathbf{Nat} ::= \mathbb{N}, one^{\mathfrak{A}} ::= 1, eq ::= =$
$zero^{\mathfrak{A}} ::= 0, plus^{\mathfrak{A}} ::= -$

➔ Filter out unwanted Σ-structures

Satisfiability Modulo Theories

- A Σ-sentence is a formula without free variables
- An axiomatic system **AX** is a set of Σ-sentences
- The Σ-theory **Th** given by **AX** is the set of all Σ-sentences implied by **AX**

A Σ-formula **F** is satisfiable modulo **Th** iff there exists a Σ-interpretation $\mathfrak{I}$ such that
- $\mathfrak{I} \models \mathbf{F}$, and
- $\mathfrak{I} \models \mathbf{G}$ for every sentence **G** in **Th**.

A Σ-formula **F** is valid modulo **Th** iff $\neg$**F** is *not* satisfiable modulo **Th**.

Exercise

- Consider the signature $\Sigma = \{\ \mathbf{Nat}, zero, one, plus, eq\ \}$,

- the theory **Th** given by the axioms

$$\forall x{:}\,\mathbf{Nat}\ (x\ eq\ x) \qquad \forall x{:}\,\mathbf{Nat}\ \forall y{:}\,\mathbf{Nat}\ \ (x\ plus\ y\ \ eq\ \ y\ plus\ x)$$

- and the formula **F** $::= \exists x{:}\,\mathbf{Nat}(\ x\ plus\ zero\ eq\ one)$.

a) Give a model witnessing that **F** is satisfiable modulo **Th**.

b) Propose an axiom that enforces that $zero$ is the neutral element of $plus$.

Solution

$F ::= \exists x\!: \mathbf{Nat}(x\ plus\ zero\ eq\ one)$

$\forall x\!: \mathbf{Nat}\ (x\ eq\ x)$ $\forall x\!: \mathbf{Nat}\ \forall y\!: \mathbf{Nat}\ (x\ plus\ y\ \ eq\ \ y\ plus\ x)$

a)

$\mathfrak{I}(x) = 1$
$\mathbf{Nat} = \mathbb{N},$
$one^{\mathfrak{A}} ::= 1,$
$zero^{\mathfrak{A}} ::= 0,$
$plus^{\mathfrak{A}} ::= +,$
$\mathrm{eq} ::= =$

✓

b)

$\mathbf{Nat} = \mathbb{N},$
$one^{\mathfrak{A}} ::= 0,$
$zero^{\mathfrak{A}} ::= 1,$
$plus^{\mathfrak{A}} ::= +,$
$\mathrm{eq} ::= =$

✗

$\forall x\!: \mathbf{Nat}\ (x\ plus\ zero\ eq\ x)$

Some important theories

- Arithmetic (with canonical axioms)
  - Presburger arithmetic: $\Sigma = \{ \mathbf{Int}, =, <, 0, 1, + \}$ **decidable**
  - Peano arithmetic: $\Sigma = \{ \mathbf{Int}, =, <, 0, 1, +, * \}$ ***un*decidable**
  - Real arithmetic: $\Sigma = \{ \mathbf{Real}, =, <, 0, 1, +, * \}$ **decidable**

- EUF: Equality logic with Uninterpreted Functions **decidable**
  - $\Sigma = \{ \mathbf{U}, =, f, g, h, \ldots \}$
  - arbitrary non-empty domain **U**
  - axioms ensure that $=$ is an equivalence relation
  - arbitrary number of uninterpreted function symbols of any arity
  - axioms do *not* constrain function symbols

- We typically need a combination of multiple theories
  - Program verification: theories for modeling different data types

Questions, murky points, feedback

https://forms.gle/Nds2CwBtEdUmR4qQ8